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A study of periodic transitions in smectic C liquid crystals

P J Barratt and B R Duffy

Department of Mathematics, University of Strathclyde, Livingstone Tower, 26 Richmond St, Glasgow G1 1XH, UK

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Abstract. We present a dynamical study of the instabilities that ensue when a relatively large magnetic field is applied suddenly across a sample of smectic liquid crystal held between parallel plates. We consider the case when the smectic layers are initially aligned parallel to the bounding plates, and the applied field is also parallel to the plates but normal to the initial average molecular orientation within the layers. Our linearized dynamical analysis for this situation confirms the prediction of static theory that there is a threshold field beyond which a Freedericksz transition occurs. Additionally, however, it is shown here that there is a second threshold beyond which a spatially periodic (rather than homogeneous) transition occurs, leading to a striped pattern; such patterns are likely to be of concern in the construction of smectic display devices. For both types of instability a bulk velocity ('backflow') is induced.

Both strong anchoring and weak anchoring are considered, and expressions are obtained for the threshold fields and for the growth rates in the various cases. Attention is focussed on the smectic C_M phase, though analogous results should be valid for more general smectic phases.

1. Introduction

The commercial development of electro-optical devices [1] has provided strong motivation for an extensive study of the director dynamics in smectic C liquid crystals over the past decade. While earlier investigations assumed that there was no coupling between flow and the director orientation, a recent paper by Leslie *et al* [2], which proposes a constrained dynamic continuum theory for smectic C liquid crystals, has enabled more recent studies to incorporate such a coupling into their analyses. In particular, backflow effects upon the switching behaviour of surface stabilized ferroelectric liquid crystal cells and the orientational relaxation in smectic C liquid crystals have been examined by Carlsson *et al* [3] and Leslie and Blake [4], respectively. Both these investigations assumed that orientation patterns after any transition are homogeneous in the directions parallel to the cell plates. However, it is well known from experience in nematic liquid crystal theory that such transition patterns may be periodic. In this event the initial growth rate of a periodic transition is greater, and hence the switching time shorter, than that associated with the corresponding homogeneous transition pattern. Hence our aim here is to investigate the possible occurrence of periodic instabilities in smectic C liquid crystals in a relatively simple experimental situation.

In this paper we examine an experimental arrangement similar to that considered by Ciaponi and Faetti [5] for nematic liquid crystals. Here a smectic C liquid crystal is confined between two large parallel plates with the smectic layers uniformly aligned everywhere parallel to the plates and subjected to a relatively large uniform magnetic field suddenly applied parallel to the smectic layers. After giving a brief outline of the continuum theory

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in section 2, we proceed to formulate the stability problem in section 3. Linearizing the continuum equations about the static equilibrium state, we seek solutions for the velocity and director fields that are spatially periodic; this reduces the mathematical problem to one of solving a pair of coupled second-order linear differential equations subject to appropriate boundary conditions. Adopting a Fourier series method of solution previously employed by Stein [6], we show that there exist critical values $h'_{\rm c}$ and $h''_{\rm c}$ of the (reduced) magnetic field strength h that mark the onset of homogeneous and periodic instabilities, respectively, in smectic $\rm C_M$ liquid crystals. We predict that (i) if $h < h'_{\rm c}$ then the initial uniform alignment is stable, (ii) if $h'_{\rm c} < h < h''_{\rm c}$ then a homogeneous transition will occur, and (iii) if $h > h''_{\rm c}$ then a periodic transition will occur. Analytic expressions are derived for these threshold fields, results for strong anchoring being given in section 4 and those for weak anchoring in section 5. The paper concludes with a brief discussion of the results in section 6.

2. The continuum equations

In this section we briefly summarize the continuum equations governing the behaviour of smectic C liquid crystals proposed by Leslie *et al* [2]. Assuming that the smectic layers consist of uniform planes with a fixed angle of tilt α between the molecular alignment and the layer normal, their constrained continuum theory introduces two orthonormal vectors to describe the smectic layered configuration. One is the unit normal $\boldsymbol{a}$ to the layers and the other is a unit vector $\boldsymbol{c}$ that is parallel to the layers and describes the direction of the tilt of the molecular alignment. In a defect-free sample $\boldsymbol{a}$ and $\boldsymbol{c}$ must satisfy the constraints

$$\boldsymbol{a}\cdot\boldsymbol{a} = \boldsymbol{c}\cdot\boldsymbol{c} = 1 \qquad \boldsymbol{a}\cdot\boldsymbol{c} = 0 \qquad \text{curl } \boldsymbol{a} = \boldsymbol{0}. \tag{2.1}$$

With the assumption that the material is incompressible the additional relevant equations for determining $\boldsymbol{a}$ and $\boldsymbol{c}$, together with the velocity vector field $\boldsymbol{v}$, are, in Cartesian tensor notation, the constraint

$$v_{i,i} = 0 \tag{2.2}$$

the linear momentum equation

$$\rho\dot{v}_i = -\tilde{p}_{,i} + \tilde{g}^a_j a_{j,i} + \tilde{g}^c_j c_{j,i} + \tilde{t}_{ij,j} \tag{2.3}$$

and the angular momentum equations

$$\left(\frac{\partial W}{\partial a_{i,j}}\right)_{,j} - \frac{\partial W}{\partial a_i} + \tilde{g}^a_i + G^a_i + \varepsilon_{ijk}\beta_{k,j} + \gamma a_i + \kappa c_i = 0 \tag{2.4}$$

and

$$\left(\frac{\partial W}{\partial c_{i,j}}\right)_{,j} - \frac{\partial W}{\partial c_i} + \tilde{g}^c_i + G^c_i + \kappa a_i + \tau c_i = 0 \tag{2.5}$$

where

$$\tilde{p} = -H_{\mathrm{m}} + p + W \qquad \tilde{t}_{ij} = \tilde{t}^{\mathrm{s}}_{ij} + \tilde{t}^{\mathrm{ss}}_{ij}$$

$$\begin{aligned}
\tilde{t}^{\mathrm{s}}_{ij} = {}& \mu_0 D_{ij} + \mu_1 a_p D^a_p a_i a_j + \mu_2 (D^a_i a_j + D^a_j a_i) \\
& + \mu_3 c_p D^c_p c_i c_j + \mu_4 (D^c_i c_j + D^c_j c_i) + \mu_5 c_p D^a_p (a_i c_j + a_j c_i) \\
& + \lambda_1 (A_i a_j + A_j a_i) + \lambda_2 (C_i c_j + C_j c_i) + \lambda_3 c_p A_p (a_i c_j + a_j c_i) \\
& + \kappa_1 (D^a_i c_j + D^a_j c_i + D^c_i a_j + D^c_j a_i) \\
& + \kappa_2 \{a_p D^a_p (a_i c_j + a_j c_i) + 2 a_p D^c_p a_i a_j\} \\
& + \kappa_3 \{c_p D^c_p (a_i c_j + a_j c_i) + 2 a_p D^c_p c_i c_j\} + \tau_1 (C_i a_j + C_j a_i) \\
& + \tau_2 (A_i c_j + A_j c_i) + 2\tau_3 c_p A_p a_i a_j + 2\tau_4 c_p A_p c_i c_j \\
\tilde{t}^{\mathrm{ss}}_{ij} = {}& \lambda_1 (D^a_j a_i - D^a_i a_j) + \lambda_2 (D^c_j c_i - D^c_i c_j) + \lambda_3 c_p D^a_p (a_i c_j - a_j c_i) \qquad (2.6) \\
& + \lambda_4 (A_j a_i - A_i a_j) + \lambda_5 (C_j c_i - C_i c_j) + \lambda_6 c_p A_p (a_i c_j - a_j c_i) \\
& + \tau_1 (D^a_j c_i - D^a_i c_j) + \tau_2 (D^c_j a_i - D^c_i a_j) + \tau_3 a_p D^a_p (a_i c_j - a_j c_i) \\
& + \tau_4 c_p D^c_p (a_i c_j - a_j c_i) + \tau_5 (A_j c_i - A_i c_j + C_j a_i - C_i a_j) \\
\tilde{g}^a_i = {}& -2(\lambda_1 D^a_i + \lambda_3 c_p D^a_p c_i + \lambda_4 A_i + \lambda_6 c_p A_p c_i \\
& + \tau_2 D^c_i + \tau_3 a_p D^a_p c_i + \tau_4 c_p D^c_p c_i + \tau_5 C_i) \\
\tilde{g}^c_i = {}& -2(\lambda_2 D^c_i + \lambda_5 C_i + \tau_1 D^a_i + \tau_5 A_i)
\end{aligned}$$

$$D^a_i = D_{ij} a_j \qquad D^c_i = D_{ij} c_j \qquad 2D_{ij} = v_{i,j} + v_{j,i}$$

$$A_i = \dot{a}_i - W_{ij} a_j \qquad C_i = \dot{c}_i - W_{ij} c_j \qquad 2W_{ij} = v_{i,j} - v_{j,i}.$$

Here ρ is the constant density, ε_{ijk} is the alternator and a superposed dot indicates a material time derivative. p, γ, τ and κ are arbitrary scalar functions and $\boldsymbol{\beta}$ is an arbitrary vector function arising from the constraints (2.1)–(2.2), while $\boldsymbol{G}^a$ and $\boldsymbol{G}^c$ denote any generalized external body forces acting and H_{m} represents the energy per unit volume due to the presence of any electric or magnetic field. Of particular importance to this paper are those forces associated with an applied magnetic field $\boldsymbol{H}$, which take the form

$$\boldsymbol{G}^a = \chi_{\mathrm{a}} (\boldsymbol{H} \cdot \boldsymbol{n}) \boldsymbol{H} \cos\alpha \qquad \boldsymbol{G}^c = \chi_{\mathrm{a}} (\boldsymbol{H} \cdot \boldsymbol{n}) \boldsymbol{H} \sin\alpha \qquad (2.7)$$

where χ_{a} denotes the anisotropic part of the magnetic susceptibility (assumed constant), and $\boldsymbol{n}$ denotes the average molecular orientation, with $\boldsymbol{n} = \boldsymbol{a}\cos\alpha + \boldsymbol{c}\sin\alpha$. Also W is the elastic stored-energy per unit volume, taking the form [2, 4]

$$\begin{aligned}
2W = {}& K^a_1 (a_{i,i})^2 + K^c_1 (c_{i,i})^2 + K^a_2 (c_i a_{i,j} c_j)^2 + K^c_2 c_{i,j} c_{i,j} + K^c_3 c_{i,j} c_j c_{i,k} c_k \\
& + 2K^a_3 a_{i,i} (c_j a_{j,k} c_k) + 2K^c_4 c_{i,j} c_j c_{i,k} a_k + 2K^{ac}_1 c_{i,i} (c_j a_{j,k} c_k) + 2K^{ac}_2 a_{i,i} c_{j,j}.
\end{aligned} \qquad (2.8)$$

The theory thus provides 16 equations (2.1)–(2.5) to determine the sixteen variables a_i, c_i, v_i, β_i, p, γ, κ and τ.

3. Formulation of the problem

We consider a sample of smectic liquid crystal confined between two horizontal flat plates of large extent, and we choose Cartesian coordinate axes so that the upper and lower plates occupy the planes $z = d$ and $z = 0$, respectively. The material is taken to be stationary initially, with the smectic layers parallel to the bounding plates and with the direction of the tilt angle initially aligned uniformly in the x-direction. This state corresponds to the simple static solution

$$\boldsymbol{v} = \boldsymbol{0} \qquad \boldsymbol{a} = (0, 0, 1) \qquad \boldsymbol{c} = (1, 0, 0) \qquad p = p_0 \tag{3.1}$$

of (2.1)–(2.5), p_0 being a constant.

We wish to consider the effect of applying a magnetic field of the form

$$\boldsymbol{H} = (0, H, 0) \tag{3.2}$$

to this arrangement, H being a prescribed constant. It is well known from the static theory that, when there is strong anchoring at the plates, the solution (3.1) becomes unstable if H is increased quasi-statically through the critical value $H_{\rm c} = (K_2^c/\chi_{\rm a})^{1/2}(\pi/d\sin\alpha)$, a Freedericksz transition then occurring. Here, however, we are interested in the situation when a magnetic field (3.2) with $H > H_{\rm c}$ is applied *suddenly* across the sample. We restrict our attention to the dynamics associated with the onset of the ensuing instability, and seek solutions of the form

$$\boldsymbol{v} = (0, \hat{v}(x, z, t), 0) \qquad \boldsymbol{c} = (1, \hat{\phi}(x, z, t), 0) \qquad \boldsymbol{a} = (0, 0, 1) \tag{3.3}$$

with $\tilde{p} = p_0 + \hat{p}$, where $\hat{v}, \hat{\phi}$ and $\hat{p}$ and their derivatives are sufficiently small that their products and powers may be neglected. The constraints (2.1) and (2.2) are satisfied identically, while a linearization of equations (2.3) and (2.5) yields $\tau = \kappa = 0$ and $\hat{p} = f(t)$, together with the equations

$$\rho\hat{v}_t = \eta_1\hat{v}_{zz} + (\kappa_1 - \tau_2 - \tau_1 + \tau_5)\hat{v}_{xz} + \eta_2\hat{v}_{xx} + \eta_3\hat{\phi}_{tx} + (\tau_1 - \tau_5)\hat{\phi}_{tz} \tag{3.4}$$

$$K_2^c\hat{\phi}_{zz} + 2K_4^c\hat{\phi}_{xz} + (K_3^c + K_2^c)\hat{\phi}_{xx} - \eta_3\hat{v}_x + (\tau_5 - \tau_1)\hat{v}_z - 2\lambda_5\hat{\phi}_t + \chi_a H^2\sin^2\alpha\hat{\phi} = 0 \tag{3.5}$$

where

$$2\eta_1 = \mu_0 + \mu_2 - 2\lambda_1 + \lambda_4 \qquad 2\eta_2 = \mu_0 + \mu_4 - 2\lambda_2 + \lambda_5 \qquad \eta_3 = \lambda_2 - \lambda_5. \tag{3.6}$$

These equations are to be solved subject to the no-slip conditions

$$\hat{v}(0) = \hat{v}(d) = 0 \tag{3.7}$$

and, for strong anchoring,

$$\hat{\phi}(0) = \hat{\phi}(d) = 0. \tag{3.8}$$

The following inequalities (given in [7] and [8]) will prove useful later:

$$\eta_1 \geqslant 0 \qquad \eta_2 \geqslant 0 \qquad \lambda_5 \geqslant 0 \qquad K_2^c \geqslant 0 \qquad K_3^c \geqslant 0. \tag{3.9}$$

It is clear from equations (3.4), (3.5) that there is no solution for which $\hat{\phi}$ and $\hat{v}$ are either purely even or purely odd functions about the plane $z = \frac{1}{2}d$. This lack of parity means that, although progress could be made in solving the problem analytically, the results would be so unwieldy that they would probably be difficult to interpret usefully. One could, of course, rely on a numerical method of solution, but this is hampered by the fact that values for the elastic and viscous material parameters have not yet been determined experimentally. (Indeed observation of transition states provides one possible means of measuring such parameters, so it would be useful to have available theoretical predictions with which the experiments may feasibly be compared.) To make progress analytically and to get some insight into the qualitative behaviour of smectic C materials we now restrict our attention to one particular subclass of these materials, namely the smectic C_M liquid crystals described by Brand and Pleiner [9]. For these materials the parameters K_4^c, K_1^{ac}, K_2^{ac}, κ_i and τ_i (for all values of i) are identically zero, so equations (3.4), (3.5) are simplified considerably.

Introducing dimensionless variables $\tilde{x}$, $\tilde{z}$, $\tilde{t}$ and V defined by

$$x = \tilde{x}\frac{d}{\pi} \qquad z = \tilde{z}\frac{d}{\pi} \qquad t = \frac{\tilde{t}\lambda_5 d^2}{K_2^c \pi^2} \qquad \hat{v} = \frac{V\pi K_2^c}{\lambda_5 d} \tag{3.10}$$

we now seek periodic solutions of the form

$$\hat{\phi} = \tilde{\phi}(\tilde{z})\exp(\mathrm{i}q\tilde{x} + s\tilde{t}) \qquad V = \mathrm{i}\tilde{v}(\tilde{z})\exp(\mathrm{i}q\tilde{x} + s\tilde{t}). \tag{3.11}$$

Here the constants q (> 0) and s are the dimensionless wavenumber and growth rate of the instability; the system is unstable if $\mathrm{Re}(s) > 0$. With inertia neglected equations (3.4) and (3.5) reduce to

$$(\mathrm{D}^2 - E_1)\tilde{v} + E_2\tilde{\phi} = 0 \tag{3.12}$$

and

$$E_3\tilde{v} + (\mathrm{D}^2 - E_4)\tilde{\phi} = 0 \tag{3.13}$$

where

$$E_1 = \frac{\eta_2}{\eta_1}q^2 \qquad E_2 = \frac{\eta_3}{\eta_1}qs \qquad E_3 = \frac{\eta_3}{\lambda_5}q \qquad E_4 = \delta q^2 + 2s - h^2$$
$$\delta = (K_2^c + K_3^c)/K_2^c \quad (> 0) \qquad h = H/H_\mathrm{c} \qquad \mathrm{D} \equiv \mathrm{d}/\mathrm{d}\tilde{z} \tag{3.14}$$

and H_c is the 'classical' Freedericksz transition threshold. For convenience the tildes will now be dropped. The relevant boundary conditions (3.7) and (3.8) (for strong anchoring) become

$$\phi(0) = \phi(\pi) = v(0) = v(\pi) = 0. \tag{3.15}$$

4. Solution for strong anchoring

Here we employ a method of solution based on Fourier series, used by Stein [6] in an analogous problem. We start by seeking a solution for v of the form

$$v = \sum_{n=1}^{\infty} a_n \sin nz \tag{4.1}$$

where the a_n are constant Fourier coefficients. This form of v satisfies the boundary conditions (3.15).

We multiply equations (3.12) and (3.13) by $(2/\pi)\sin nz$, integrate by parts with respect to z from $z = 0$ to $z = \pi$, and use (3.15) to simplify the results. This yields the linear algebraic equations

$$-(n^2 + E_1)a_n + E_2 I_n = 0 \tag{4.2}$$

$$E_3 a_n - (n^2 + E_4)I_n = 0 \tag{4.3}$$

where

$$I_n := \frac{2}{\pi}\int_0^{\pi} \phi(z)\sin nz \,\mathrm{d}z.$$

For a non-trivial solution of these equations we require

$$(n^2 + E_1)(n^2 + E_4) - E_2 E_3 = 0$$

from which we deduce that the growth rate $s = s_n(q)$ is given by

$$s_n(q) = (n^2 + \sigma_1 q^2)(h^2 - n^2 - q^2\delta)\big/2[n^2 + (\sigma_1 - \sigma)q^2] \tag{4.4}$$

for $n = 1, 2, 3, \ldots$, where

$$\sigma := \frac{\eta_3^2}{2\eta_1\lambda_5} \quad (>0) \qquad \sigma_1 := \frac{\eta_2}{\eta_1} \quad (>0) \tag{4.5}$$

(the inequalities here being a consequence of (3.9)). Equation (4.4) gives the growth rate s_n as a function of q for a given reduced magnetic field strength h and a given mode number n; clearly s_n is purely real for all q, so there is no temporal oscillation at the instability. Also, it follows that

$$s_n(0) = \tfrac{1}{2}(h^2 - n^2) \qquad \frac{\mathrm{d}s_n(0)}{\mathrm{d}q} = 0 \qquad \frac{\mathrm{d}^2 s_n(0)}{\mathrm{d}q^2} = \frac{h^2\sigma - n^2\sigma - n^2\delta}{n^2} \tag{4.6}$$

which shows that the smallest field h for which $s_n(0)$ becomes positive is $h = 1$ (corresponding to the mode $n = 1$); therefore the threshold h_c' for a homogeneous ($q = 0$) instability is given by

$$h_c' = 1. \tag{4.7}$$

Thus a dynamical linear stability analysis confirms our previous statement, based on the static theory, that the system is stable until H exceeds the critical field H_c.

Also, we note from (4.6) that $q = 0$ is a stationary point of the $s_n(q)$ curve, a maximum if $h < h_n$ and a minimum if $h > h_n$, where $h_n^2 = n^2(\sigma + \delta)/\sigma$. Thus if $h > h_n$ then $s_n(q)$ is larger for some non-zero q than it is for $q = 0$, so a periodic transition (rather than a homogeneous transition) must occur. This suggests that the threshold field h_c'' for a periodic transition is the smallest value of h_n, i.e.

$$h_c'' = \left(1 + \frac{\delta}{\sigma}\right)^{\frac{1}{2}} = \left[1 + \frac{2\eta_1\lambda_5}{\eta_3^2}\left(1 + \frac{K_3^c}{K_2^c}\right)\right]^{\frac{1}{2}}. \tag{4.8}$$

The above argument, based on behaviour near $q = 0$, can be made more convincing, as follows. First, using the fact that $\sigma_1 > \sigma$ (see the appendix), one may show that for $h < h_n$ the $s_n(q)$ curve has a single stationary point, a maximum, at $q = 0$, but for $h > h_n$ this curve has two stationary points in $q \geqslant 0$, a minimum at $q = 0$ and a (global) maximum at $q = q_{\max}$, where

$$q_{\max} = \frac{n}{(\sigma_1 - \sigma)^{\frac{1}{2}}}\left\{\left[1 + \frac{\sigma(\sigma_1 - \sigma)(h^2 - h_n^2)}{\sigma_1 n^2 \delta}\right]^{\frac{1}{2}} - 1\right\}^{\frac{1}{2}}. \tag{4.9}$$

Secondly, the difference between the growth rates $s_m(q)$ and $s_n(q)$ of two modes m and n is given by

$$s_m(q) - s_n(q) = -\tfrac{1}{2}(m^2 - n^2)Q_{mn} \tag{4.10}$$

where

$$Q_{mn} := 1 + \frac{\sigma q^2[(\sigma_1 - \sigma)q^2 + h^2 - q^2\delta]}{[m^2 + (\sigma_1 - \sigma)q^2][n^2 + (\sigma_1 - \sigma)q^2]}.$$

Now for these modes to be unstable equation (4.4) shows that $h^2 > q^2\delta$, in which case $Q_{mn} > 0$. Therefore, for any value of q, a 'lower' mode corresponds to a larger growth rate; this means that when seeking a threshold field or a maximum growth rate we need consider only the lowest mode, $n = 1$. Thus overall we conclude that when $h > h_n$ the $n = 1$ mode with wavenumber $q = q_{\max}$ will have the largest growth rate, and that the threshold field h_c'' is indeed as in (4.8). Given equation (4.7) and the inequalities (3.9) we clearly have

$$h_c'' > h_c' \tag{4.11}$$

so for field strengths h that exceed h_c'' we anticipate that a periodic rather than a homogeneous transition will occur, the wavelength of the expected stripe pattern being proportional to q_{max}^{-1} with $n = 1$.

5. Weak anchoring

In the event that there is weak anchoring at the two plane boundaries, the theory for nematics suggests that the strong anchoring conditions given in (3.8) should be replaced by

$$\hat{\phi} - b\hat{\phi}_z = 0 \qquad \hat{\phi} + b\hat{\phi}_z = 0 \tag{5.1}$$

on $z = 0$ and $z = \pi$, respectively. Here b is an extrapolation length [10] and we assume that this anchoring constant is the same for the two boundaries. Introducing the dimensionless variables (3.10) and seeking periodic solutions of the form (3.11), we again reduce the problem to that of solving the differential equations (3.12) and (3.13), but now subject to the boundary conditions

$$\phi(0) - b_1\phi_z(0) = 0 \qquad \phi(\pi) + b_1\phi_z(\pi) = 0 \qquad v(0) = v(\pi) = 0 \tag{5.2}$$

where $b_1 = b\pi/d$. Seeking a solution for v of the form (4.1) we again multiply (3.12) and (3.13) by $(2/\pi)\sin nz$ and integrate from $z = 0$ to $z = \pi$. This leads to the linear algebraic system of equations

$$-(n^2 + E_1)a_n + E_2 I_n = 0 \tag{5.3}$$

$$E_3 a_n - (n^2 + E_4)I_n + R_n = 0 \tag{5.4}$$

where

$$R_n := (2n/\pi)\{\phi(0) - (-1)^n\phi(\pi)\}. \tag{5.5}$$

Eliminating I_n we obtain

$$a_n = R_n E_2/\Delta_n \qquad \Delta_n := (n^2 + E_1)(n^2 + E_4) - E_2 E_3. \tag{5.6}$$

From equations (3.13) and (4.1) ϕ is given by

$$\phi = \sum_{n=1}^{\infty} \frac{E_3 a_n \sin nz}{n^2 + E_4} + B_1 \cos c(z - \tfrac{1}{2}\pi) + B_2 \sin c(z - \tfrac{1}{2}\pi) \tag{5.7}$$

where B_1 and B_2 are arbitrary constants, and $c^2 = -E_4$. With equation (5.7) the boundary conditions (5.2) yield two equations from which we obtain

$$B_1[\cos(\tfrac{1}{2}c\pi) - b_1 c \sin(\tfrac{1}{2}c\pi)] = b_1 E_3 \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{n a_n}{n^2 + E_4} \tag{5.8}$$

$$B_2[\sin(\tfrac{1}{2}c\pi) + b_1 c \cos(\tfrac{1}{2}c\pi)] = -b_1 E_3 \sum_{\substack{n=1 \\ (n\ \text{even})}}^{\infty} \frac{n a_n}{n^2 + E_4}. \tag{5.9}$$

However, from equations (5.5)–(5.7) we also have

$$a_n = \frac{4E_2 n}{\pi\Delta_n} \begin{cases} B_1 \cos\frac{1}{2}c\pi & (n\ \text{odd}) \\ -B_2 \sin\frac{1}{2}c\pi & (n\ \text{even}) \end{cases} \tag{5.10}$$

which, with (5.8) and (5.9), leads to the consistency conditions

$$\tfrac{1}{4}\pi[1 - b_1 c \tan(\tfrac{1}{2}c\pi)] = b_1 E_2 E_3 \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{n^2}{(n^2 + E_4)\Delta_n} \tag{5.11}$$

$$\tfrac{1}{4}\pi[1 + b_1 c \cot(\tfrac{1}{2}c\pi)] = b_1 E_2 E_3 \sum_{\substack{n=1 \\ (n\ \text{even})}}^{\infty} \frac{n^2}{(n^2 + E_4)\Delta_n}. \tag{5.12}$$

For given values of h and b_1, these equations determine the growth rates s as functions of q (or of q^2, to be more precise) for different modes. Equations (5.11) and (5.12) are associated with modes that are respectively symmetric and anti-symmetric about the plane $z = \frac{1}{2}\pi$.

To determine thresholds we consider the behaviour near $q = 0$. We expand for small q by writing

$$s = s_0 + s_2 q^2 + \cdots \tag{5.13}$$

where s_0, $s_2, \ldots$ depend on h and b_1. With only terms to $\mathrm{O}(q^2)$ retained, equations (5.11) and (5.12) become

$$1 - b_1 k \tan\left(\tfrac{1}{2}\pi k\right) + \frac{b_1 q^2 (2s_2 + \delta)}{4k} \left\{2\tan\left(\tfrac{1}{2}\pi k\right) + \pi k \sec^2\left(\tfrac{1}{2}\pi k\right)\right\}$$
$$= \frac{8 b_1 \sigma q^2 s_0}{\pi} \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{1}{(n^2 - k^2)^2} \tag{5.14}$$

$$1 + b_1 k \cot\left(\tfrac{1}{2}\pi k\right) - \frac{b_1 q^2 (2s_2 + \delta)}{4k} \left\{2\cot\left(\tfrac{1}{2}\pi k\right) - \pi k \operatorname{cosec}^2\left(\tfrac{1}{2}\pi k\right)\right\}$$
$$= \frac{8 b_1 \sigma q^2 s_0}{\pi} \sum_{\substack{n=1 \\ (n\ \text{even})}}^{\infty} \frac{1}{(n^2 - k^2)^2} \tag{5.15}$$

where

$$k^2 = h^2 - 2s_0. \tag{5.16}$$

Equating terms of order q^0 in these respectively yields

$$b_1 k \tan\left(\tfrac{1}{2}\pi k\right) = 1 \qquad b_1 k \cot\left(\tfrac{1}{2}\pi k\right) = -1. \tag{5.17}$$

At marginal stability we have $s_0 = 0$, and the field is $h = h'$ (say), so that $k = h'$ and

$$b_1 h' \tan\left(\tfrac{1}{2}\pi h'\right) = 1 \qquad b_1 h' \cot\left(\tfrac{1}{2}\pi h'\right) = -1. \tag{5.18}$$

There are infinitely many solutions h' of these equations, and the threshold field h'_{c} for a homogeneous transition is the smallest positive such solution. One can show easily that the values of h' from (5.18) satisfy $2M \leqslant h' \leqslant 2M + 1$ and $2M + 1 \leqslant h' \leqslant 2M + 2$, respectively, where $M = 0, 1, 2, \ldots$; clearly therefore the threshold field is associated with the 'lowest' ($M = 0$) symmetric mode, and h'_{c} is that solution of the equation

$$b_1 h'_{\mathrm{c}} \tan\left(\tfrac{1}{2}\pi h'_{\mathrm{c}}\right) = 1 \tag{5.19}$$

that satisfies

$$0 \leqslant h'_{\mathrm{c}} \leqslant 1. \tag{5.20}$$

In particular, in the limit $b_1 \to 0$ we recover from (5.19) the expected strong-anchoring threshold $h'_c = 1$, and for very weak anchoring ($b_1 \to \infty$) we have

$$h'_c \sim \left(2/\pi b_1\right)^{\frac{1}{2}} \to 0. \tag{5.21}$$

These results are analogous to the corresponding 'classical' results for the case of a nematic material with weak anchoring.

We now consider the possibility of a periodic transition when h exceeds h'_c and s_0 is non-zero. With the identities

$$\sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{1}{(n^2-k^2)^2} = \frac{\pi}{16k^3}\left\{\pi k \sec^2\left(\tfrac{1}{2}\pi k\right) - 2\tan\left(\tfrac{1}{2}\pi k\right)\right\}$$

$$\sum_{\substack{n=1 \\ (n\ \text{even})}}^{\infty} \frac{1}{(n^2-k^2)^2} = \frac{\pi}{16k^3}\left\{\pi k \operatorname{cosec}^2\left(\tfrac{1}{2}\pi k\right) + 2\cot\left(\tfrac{1}{2}\pi k\right) - \left(8/\pi k\right)\right\}$$

and, with b_1 eliminated by means of (5.17), from the $O(q^2)$ terms in (5.14) and (5.15) we find that

$$s_0 = \frac{(2s_2+\delta)k^2}{2\sigma}\begin{cases} f_o(\pi k) & (n\ \text{odd}) \\ f_e(\pi k) & (n\ \text{even}) \end{cases} \tag{5.22}$$

where

$$f_o(\xi) = \frac{\xi + \sin\xi}{\xi - \sin\xi} \qquad f_e(\xi) = \frac{\xi - \sin\xi}{\xi + \sin\xi - 4\xi^{-1}(1-\cos\xi)}. \tag{5.23}$$

Now at the margin of a periodic instability

$$s_2 = 0 \qquad h = h'' \quad \text{(say)} \tag{5.24}$$

and from equations (5.16)–(5.18) we also have

$$k = h' \qquad 2s_0 = h^2 - (h')^2. \tag{5.25}$$

With equation (5.22) these lead to

$$h'' = h'\left\{1 + (\delta/\sigma) f_o(\pi h')\right\}^{\frac{1}{2}} \qquad (2M \leqslant h' \leqslant 2M+1) \tag{5.26}$$

for the symmetric modes, and to

$$h'' = h'\left\{1 + (\delta/\sigma) f_e(\pi h')\right\}^{\frac{1}{2}} \qquad (2M+1 \leqslant h' \leqslant 2M+2) \tag{5.27}$$

for the anti-symmetric modes (with $M = 0, 1, 2, \ldots$ in both). It is easy to show (for example, simply by plotting $f_o(\xi)$ and $f_e(\xi)$ over the appropriate ξ-domains) that the lowest value h''_c of h'' obtainable from (5.26) and (5.27) comes from (5.26) (that is, a symmetric mode) with $M = 0$. Thus the threshold field h''_c is given by

$$h''_c = h'_c\left\{1 + \frac{\delta}{\sigma}\left[\frac{\pi h'_c + \sin\pi h'_c}{\pi h'_c - \sin\pi h'_c}\right]\right\}^{\frac{1}{2}} \qquad (0 \leqslant h'_c \leqslant 1) \tag{5.28}$$

with h'_c determined by (5.19) and (5.20). It is clear from this that equation (4.11) again holds (it being known that $\delta/\sigma > 0$). Thus when h exceeds the critical value h''_c we expect the resulting transition to be periodic.

In the limit $b_1 \to 0$ we recover from (5.28) the strong-anchoring result for h''_c, as given in (4.8). Also as $b_1 \to \infty$ we have

$$h''_c \to \left(\frac{12\delta}{\pi^2\sigma}\right)^{\frac{1}{2}} = \left\{\frac{24\eta_1\lambda_5}{\pi^2\eta_3^2}\left(\frac{K_2^c + K_3^c}{K_2^c}\right)\right\}^{\frac{1}{2}}. \tag{5.29}$$

Incidentally, equation (5.28) may be written as

$$h''_c = \left\{(h'_c)^2 + \frac{\delta}{\sigma}\left[\frac{\left(\pi h'_c\right)^2 f_0(\pi h'_c)}{\pi^2}\right]\right\}^{\frac{1}{2}} \tag{5.30}$$

and it is found that the quantity in square brackets here varies only weakly with h'_c in the interval $0 \leqslant h'_c \leqslant 1$, decreasing monotonically from the value $12/\pi^2$ (≈ 1.216) at $h'_c = 0$ to the value 1 at $h'_c = 1$. Thus a crude but fairly accurate approximation to (5.28) over the whole interval $0 \leqslant b_1 < \infty$ is

$$h''_c \approx \left\{(h'_c)^2 + \epsilon(\delta/\sigma)\right\}^{\frac{1}{2}} \qquad \epsilon \approx 1. \tag{5.31}$$

6. Concluding remarks

We have obtained analytic expressions, as given by equations (4.7), (4.8), (5.19) and (5.28), for the critical reduced threshold fields for the onset of both homogeneous and periodic transient patterns in smectic C_M liquid crystals when either weak or strong anchoring obtains at the boundaries. For a sudden application of a magnetic field across the sample we anticipate stability for $h < h'_c$, a homogeneous transition for $h'_c < h < h''_c$, and a periodic transition for $h > h''_c$. The results presented here parallel those found by Ciaponi and Faetti [5] for an analogous problem in nematics.

Although the above quantitative analysis is applicable only for smectic C_M materials, it is reasonable to anticipate that the qualitative behaviour described (that is, the existence of a second threshold h''_c in addition to the classical Freederickzs threshold h'_c above which periodic transition patterns should be observed) will also be pertinent to smectic C materials. Of course, since the analysis presented here is linear, we cannot say how the initial instability will develop. However, the possibility of periodic transitions for sufficiently large magnetic fields suggests that they may be of some significance in the effects of backflow in smectic C liquid crystals. As a final observation, we note that in the linear analysis presented here, there are no permeation effects in this problem for either smectic C_M or smectic C liquid crystals.

Appendix. An inequality concerning the viscosity coefficients

By equations (3.9) we have $\sigma > 0$ and $\sigma_1 > 0$. Using the viscous dissipation inequality

$$\tilde{t}^s_{ij} D_{ij} - \tilde{g}^a_i A_i - \tilde{g}^c_i C_i \geqslant 0 \tag{A.1}$$

(see [8]) we show further that

$$\sigma_1 > \sigma > 0. \tag{A.2}$$

Writing $\boldsymbol{a} = \boldsymbol{i}$ and $\boldsymbol{c} = \boldsymbol{j}$ we have $\boldsymbol{A} = A_2\boldsymbol{j} + A_3\boldsymbol{k}$ and $\boldsymbol{C} = -A_2\boldsymbol{i} + C_3\boldsymbol{k}$ for some A_2, A_3, C_3 (so that $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{A}$ and $\boldsymbol{C}$ automatically satisfy the equations $\boldsymbol{a}\cdot\boldsymbol{c} = 0$, $\boldsymbol{a}\cdot\boldsymbol{A} = 0$, $\boldsymbol{c}\cdot\boldsymbol{C} = 0$, and $\boldsymbol{A}\cdot\boldsymbol{c} + \boldsymbol{a}\cdot\boldsymbol{C} = 0$). Then equation (A.1) gives

$$(2\mu_0 + \mu_1 + 2\mu_2)D_{11}^2 + (2\mu_0 + \mu_3 + 2\mu_4)D_{22}^2 + 2(\mu_0 + \mu_2 + \mu_4 + \mu_5)D_{12}^2$$

$$+2\mu_0 D_{11} D_{22} + 2(\mu_0+\mu_2) D_{13}^2 + 2(\mu_0+\mu_4) D_{23}^2 + 4(\lambda_1-\lambda_2+\lambda_3) A_2 D_{12}$$
$$+4\lambda_1 A_3 D_{13} + 4\lambda_2 C_3 D_{23} + 2(\lambda_4+\lambda_5+\lambda_6) A_2^2 + 2\lambda_4 A_3^2 + 2\lambda_5 C_3^2 \geqslant 0 \tag{A.3}$$

D_{33} having been eliminated by use of the incompressibility condition $D_{11}+D_{22}+D_{33}=0$. From the terms in D_{23} and C_3 in (A.3) we deduce that

$$\lambda_5(\mu_0+\mu_4) > \lambda_2^2. \tag{A.4}$$

However, $\sigma_1 - \sigma$ may be written as

$$\sigma_1 - \sigma = \left[\lambda_5(\mu_0+\mu_4) - \lambda_2^2\right] / 2\eta_1\lambda_5,$$

and this, with (3.9) and (A.4), leads to (A.2), as required.

Two further observations lend support to (A.2). First, if (A.2) were *not* satisfied then the strong-anchoring growth rate $s_n(q)$ for mode n in (4.4) would become infinite when $q = n/(\sigma-\sigma_1)^{1/2}$, which seems unphysical.

Secondly Carlsson *et al* [8] have noted that, at least for the case when the tilt angle α is small, there exist simple correspondences

$$\begin{array}{lll} \alpha_1 \to \mu_3 & \alpha_2 \to \lambda_2 - \lambda_5 & \alpha_3 \to \lambda_2 + \lambda_5 \\ \alpha_4 \to \mu_0 & \alpha_5 \to \mu_4 - \lambda_2 & \alpha_6 \to \mu_4 + \lambda_2 \end{array} \tag{A.5}$$

between the viscosities $\alpha_1, \alpha_2, \ldots, \alpha_6$ of a *nematic* material and those of a smectic material (see table II of [8]). These correspondences may be combined with the known inequality

$$(\alpha_3-\alpha_2)(2\alpha_4+\alpha_5-\alpha_6) > (\alpha_2+\alpha_3)^2$$

(see equation (35) of [11]) to give (A.4) again, and hence (A.2).

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